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On a Moving Server with Bounded Range

Carole Siegel

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ABSTRACT

An extension of a model considered by Hanisch and Hirsch [Comm. Pure Appl. Math., 11/63] is investigated in which the server is of "bounded range." The properties of $p^{(1)}(b,B)$, a measure of effectiveness of the server relative to the system, are explored. In particular conditions are given for the case of an ineffective server, ($p^{(1)}(b,B) = 0$). The model and its properties are further extended to a "partial absorbing barrier."

1. INTRODUCTION AND MATHEMATICAL MODEL

In this paper we consider the following moving server problem. Members of an infinite queue moving with uniform speed towards a barrier are serviced consecutively by a server moving along with the items. The first item on queue is b units from the barrier. After service on an item is completed, the server begins service on the next element of the queue without loss of time if and only if the item at that point is $\leq B$ units from the barrier. If the item is $> B$ units from the barrier, service begins when the item is B units from the barrier. The barrier is absorbing in that the process terminates if an item reaches the barrier without service on it being completed. We refer to this process as a moving server process with bounded range, B. Extensive results have been obtained by H. Hanisch and W. Hirsch [3] for the case of unbounded range.

This process serves as a model for a missile battery of finite range, B. The items are a file of planes moving toward an objective — a missile battery. The battery is able to fire on (service) a plane when it is at a distance $\leq B$ from it. If a plane reaches the battery without being destroyed, it destroys the battery.

The mathematical model is as follows. Let $E_1, E_2, \ldots$ denote respectively the members of an infinite queue where the subscript indicates the order in line. We think of these elements as being located initially at points of the set $R^+ = \{x: 0 \leq x < \infty\}$; the spacing between adjacent elements being stochastically determined. Let b denote the distance of E_1 at time $t = 0$ from the origin. Let η_j denote the distance between E_j and E_{j+1}. η_j is a finite valued random variable with distribution function

$$P\{\eta_j \leq x\} = G(x) , \qquad j = 1,2,\dots .$$

Let ξ_j denote the time required for completion of service on E_j with

$$P\{\xi_j \leq x\} = F(x) , \qquad j = 1,2,\dots .$$

ξ_j may, with positive probability, be infinite. The random variables $\{\xi_j, \eta_j\}_{j=1}^{\infty}$ are all assumed to be mutually independent.

The queue moves with unit velocity toward the origin, its elements being served successively. If E_j is $\leq B$ units from the origin, service on E_j is initiated instantaneously upon completion of service on E_{j-1}. If E_j is $> B$ units from the origin, service on E_j does not begin until E_j is B units from the barrier. If an element E_j arrives at 0 before service on it has been completed, the process terminates.

Let $p^{(1)}(b,B)$ denote the probability of serving infinitely many elements in the queue. If infinitely many elements in the queue are serviced this implies the server survives forever. We consider then $p^{(1)}(b,B)$ to be a measure of the effectiveness of the service facility. In Section 2 we exploit known results of the case of unbounded range to investigate properties of p in terms of the parameters of the system. In Section 3, using ergodic theorems of Markov processes, we sharpen results of Section 2. In Section 4, we extend the results obtained to a model in which we consider the process to terminate with probability α, $0 \le \alpha \le 1$ if an item reaches the barrier and to continue with probability $1-\alpha$. We refer to this as an (α,b,B) process.

2. RECURRENCE RELATION AND BASIC LEMMAS FOR (1,b,B) PROCESS

Let ν denote the number of elements served before termination of the process. Let

$$\begin{aligned} p(\nu = n \mid E_1 \text{ at } b) &= p_n^{(1)}(b,B) , && b \leq B \\ &= p_n^{(1)}(B,B), && b > B . \end{aligned}$$

$\nu = 0$ if E_1 arrives at the origin unserviced. Hence

$$\begin{aligned} p_0^{(1)}(b,B) &= 1 - F(b) , && b \leq B \\ &= p_0^{(1)}(B,B) , && b > B . \end{aligned}$$

ν takes the value $n \geq 1$ if and only if the event service on E_1 is completed and exactly n-1 additional elements are serviced prior to termination. Denoting this event by A we have,

$$\begin{aligned} p(A \mid \xi_1 = t,\ \eta_1 = x) &= p_{n-1}^{(1)}(b-t+x,B) , && x \leq B-(b-t) \\ &= p_{n-1}^{(1)}(B,B) , && x > B-(b-t) \end{aligned}$$

We are thus led to

$$(1) \quad p_n^{(1)}(b,B) = \int_{[0,b)} dF(t) \Big[\int_{[0,B+t-b)} p_{n-1}^{(1)}(b-t+x,B)\, dG + (1 - G(B - (b-t)))\, p_{n-1}^{(1)}(B,B) \Big] .$$

Let $\pi_k^{(1)}(b,B)$ denote the probability of servicing at least k items in the queue before absorption. We have the relationship

$$(2) \qquad \lim_{k\to\infty} \pi_k^{(1)}(b,B) = p^{(1)}(b,B) .$$

Furthermore, it can be argued that $\pi_k^{(1)}(b,B)$ satisfies the recurrence relation (1) with initial condition

$$\pi_0^{(1)}(b,B) = 1 .$$

Let $\pi_k^{(1)}(b,\infty)$, $p^{(1)}(b,\infty)$, etc. be the associated quantities in the (1, b) process of unbounded range. Comparison of these probabilities with those of the case of bounded range enable us to obtain some immediate extensions of results of the unbounded range.

Lemma 1. $\pi_k^{(1)}(b,\infty) \geq \pi_k^{(1)}(b,B)$, $k = 0,1,2,\ldots$.

Proof. $\pi_{k+1}^{(1)}(b,\infty)$ satisfies the following recurrence relation with $\pi_0^{(1)}(b,\infty) = 1$. We have

$$\begin{aligned}
\pi_{k+1}^{(1)}(b,\infty) &= \int_{[0,b)} dF(t) \int_{[0,\infty)} \pi_k^{(1)}(b-t+x,\infty)\, dG(x) \\
(3) \qquad &= \int_{[0,b)} dF(t) \int_{[0,B+t-b)} \pi_k^{(1)}(b-t+x,\infty)\, dG(x) \\
&\quad + \int_{[0,b)} dF(t) \int_{[B+t-b,\infty)} \pi_k^{(1)}(b-t+x,\infty)\, dG(x) .
\end{aligned}$$

Certainly it is true that

$$\pi_0^{(1)}(b,\infty) = 1 \geq \pi_0^{(1)}(b,B) = 1 ,$$

and let us assume for some fixed k > 0 that

$$(4) \qquad \pi_k^{(1)}(b,\infty) \geq \pi_k^{(1)}(b,B) , \qquad b \leq B .$$

H. Hanisch and W. Hirsch [3] have shown that $\pi_k^{(1)}(b,\infty)$ is monotone nondecreasing in b. Hence

$$(5) \qquad \pi_k^{(1)}(b-t+x,\infty) \geq \pi_k^{(1)}(B,\infty) , \qquad x \geq B+t-b .$$

Under assumption (4), and using (5) and (3), we get

$$\begin{aligned}\pi_{k+1}^{(1)}(b,\infty) \geq & \int_{[0,b)} dF(t) \int_{[0,B+t-b)} \pi_k^{(1)}(b-t+x,B)\, dG(x) \\ & + \int_{[0,b)} dF(t) \int_{[B+t-b,\infty)} \pi_k^{(1)}(B,B)\, dG(x) \\ = & \ \pi_{k+1}^{(1)}(b,B) .\end{aligned}$$

Hence Lemma 1 is true by the principle of mathematical induction.

Corollary 1.1.

$$(6) \qquad p^{(1)}(b,\infty) \geq p^{(1)}(b,B) .$$

Proof: Noting that (2) holds as well for $\pi_k^{(1)}(b,\infty)$, the proof follows from Lemma 1.

Theorem 1. If $E(F) \geq E(G)$, $E(G) < \infty$, and F and G do not both degenerate at the same point, then

$$p^{(1)}(b,B) \equiv 0 \ . \tag{7}$$

Proof: H. Hanisch and W. Hirsch have shown under the hypothesis of this theorem that $p^{(1)}(b,\infty) \equiv 0$ (7) follows from (6).

If $E(G) > E(F)$, and the range of the server is unbounded, then Hanisch and Hirsch have shown

$$\lim_{b \to \infty} p^{(1)}(b,\infty) = 1 \ .$$

If the bound $B < \infty$, then it is not necessarily true that $p^{(1)}(B,B) = 1$. Elementary examples can be constructed to illustrate this.

3. THE MARKOVIAN CHARACTER OF THE PROCESS

It is possible by introducing an auxiliary random variable to treat the (1,b,B) process as Markovian. By exploiting this fact, we are able to obtain a result which sharpens Theorem 1 and which is not true for unbounded range. We employ the terminology of Chung [1] and Doob [2] in this treatment. Let $x_n(\omega)$ be the distance of the item E_n from the barrier at the time it becomes lead item in the queue. ($x_n(\omega)$ is defined to be zero if the nth item never becomes the lead element.) x_n can be defined recursively as follows

$$
\begin{array}{ll}
x_1 = b & \\
x_n = x_{n-1} - (\xi_{n-1} - \eta_{n-1}) & \left.\begin{array}{l} x_{n-1} > 0\ ,\ x_{n-1} > \xi_{n-1} \\ x_{n-1} - \xi_{n-1} + \eta_{n-1} \leq B \end{array}\right\} \\
 = B & \left.\begin{array}{l} x_{n-1} > 0,\ x_{n-1} > \xi_{n-1} \\ x_{n-1} - \xi_{n-1} + \eta_{n-1} > B \end{array}\right\} \\
 = 0 & \left.\begin{array}{l} x_{n-1} > 0\ ,\ x_{n-1} \leq \xi_{n-1} \end{array}\right\} \\
 = 0 & \left.\begin{array}{l} x_{n-1} = 0\ . \end{array}\right\}
\end{array}
$$

The definition of $\{x_n\}_{n=1}^{\infty}$ implies it is a Markov process since x_n is defined in terms of x_{n-1}, ξ_{n-1} and η_{n-1} and

(ξ_{n-1}, η_{n-1}) is independent of $(x_1, x_2, \ldots, x_{n-1})$. Note that we have defined the process so that service at the origin is impossible.

3.1 Discrete time

The state space for discrete time becomes the set 0,1,2,...,B. (If we allow service at the origin, we would introduce an artificial state, -1.) Thus for discrete time, we have a Markov chain which moreover is stationary since the items E_n have equal service time distributions and equal spacing distributions. Let π_{ij} denote the transition probability from state i to j.

$$\pi_{ij} = P(x_n(\omega) = j \mid X_{n-1}(\omega) = i)$$

0 is an absorbing state in that if $x_n(\omega) = 0$, $x_m(\omega) = 0$, $m \geq n$. We note that

$$(8) \qquad \sum_{n=1}^{\infty} P(x_n(\omega) = 0) = 1 - p^{(1)}(b,B) .$$

In what follows, we sharpen Theorem 1 using ergodic theorems of Markov processes. For a finite Markov chain, we have the known

Theorem 2. In a finite Markov chain with stationary transition probabilities, the probability of the system staying forever in a nonrecurrent class is 0.

We first have

Theorem 3. If F and G are discrete and $F(B) < 1$, $p^{(1)}(b,B) \equiv 0$.

Proof: j leads to 0 since

$$\pi_{j0} = 1 - F(j-) \geq 1 - F(B-) > 0 .$$

Now $\pi_{0k} = 0$, $k > 0$. It follows that $\pi_{0j}^{(n)} = 0$ for all n; hence j is inessential and therefore nonrecurrent, $0 < j \leq B$. We have that $\pi_{00}^{(n)} = 1$ for all n. Hence 0 is the only recurrent state. From Theorem 2 and (8), we obtain $p^{(1)}(b,B) \equiv 0$.

Theorem 4. If F and G are discrete and $F(B) = 1$ and $F(b) \neq 1$, $b \leq B$, and if

$$P(\xi(\omega) - \eta(\omega) > 0) > 0 ,$$

then

$$p^{(1)}(b,B) \equiv 0 .$$

Proof: The hypothesis implies there exists a $b^* < B$ such that

$$(9) \qquad \begin{aligned} \pi_{j0} &> 0 , \qquad j < b^* \\ &= 0 , \qquad j \geq b^* \end{aligned} .$$

(9) implies the states $0 < j < b^*$ are inessential since

$\pi_{0k}^{(n)} = 0$, $k > 0$ and for all n. Consider $b \geq b^*$. The event

$$\{x_1(\omega) = b,\ \xi_1 > \eta_1\} \subseteq \{x_1(\omega) = b, x_2(\omega)=k, \text{ for some } k < b\}$$

and hence

$$(10) \qquad P(x_1(\omega) = b)P(\xi_1 > \eta_1) \leq P(x_1(\omega) = b) \sum_{k=0}^{b-1} \pi_{bk} .$$

Let us suppose the chain starts at b so that $P(x_1(\omega) = b) > 0$. Then from (10) we see there exists a $k_1 < b$ for which $\pi_{bk_1} > 0$. The event

$$\{x_1(\omega) = b, x_2(\omega)=k_1,\ \xi_2 > \eta_2\} \subseteq \{x_1(\omega)=b,\ x_2(\omega)= k_1,\ x_3(\omega)=k \text{ for some } k < k_1\}$$

and thus

$$0 < P(x_1 = b)\pi_{bk_1}\ P(\xi_2 > \eta_2) \leq P(x_1 = b)\pi_{bk_1} \sum_{k=0}^{k_1-1} \pi_{k_1 k}$$

Hence there exists a k_2 such that $\pi_{k_1 k_2} > 0$. Repeating the argument a finite number of times, say n, we will arrive at 0. That is,

$$\pi_{b0}^{(n)} \geq \pi_{bk_1}\ \pi_{k_1 k_2} \cdots \pi_{k_n 0} > 0 .$$

Hence b is inessential and, therefore, nonrecurrent, $0 < b \leq B$. Now $\pi_{00}^{(n)} = 1$ for all n. 0 is the only recurrent state. By Theorem 2 and (8), $p^{(1)}(b,B) \equiv 0$.

Note that this theorem implies even if $E(G) > E(F)$, provided $P(\xi - \eta > 0) > 0$, that $p^{(1)}(b,B) \equiv 0$ — a result not true for unbounded range.

3.2 Continuous time

If the distributions F and G are arbitrary, the state space of the discrete parameter Markov process $\{x_n\}$ becomes the closed interval $X = [0,B]$. Let $\mathcal{F}_X$ be the linear Borel field over $[0,B]$. We define the stochastic transition function $\pi(\cdot,\cdot)$ of $u \in X$ and $A \in \mathcal{F}_X$ as follows. Let $A = (0,v)$. Then

$$(11)\; P(x_{m+1} \in A \mid x_m = u) = \pi(u,A)$$

$$= \int_{(u-v,u)} dF(t)\, G([v-(u-t)]-)\,, \qquad 0 < v < u \leq B$$

$$= \int_{[0,u)} dF(t)\, G([v-(u-t)]-)\,, \qquad B > v \geq u \geq 0$$

$$\pi(u,[0,0]) = 1 - F(u-)$$

$$\pi(u,[B,B]) = \int_{[0,u)} dF(t)(1 - G([B - (u-t)] -))\,.$$

We extend $\pi(u,\cdot)$ to all sets of $\mathcal{F}_X$ in the standard way. $\pi(u,A)$ for fixed u will determine a probability measure in A and for fixed A will be a u function measurable with respect to the field $\mathcal{F}_X$. The n step transition probabilities are calculated from

$$\pi^{(n+1)}(u,A) = \int_X \pi^{(n)}(v,A)\, \pi(u,dv) \, , \quad \pi(u,A) = \pi^{(1)} .$$

To extend Theorem 2 utilized on the discrete time case, the following hypothesis of Doeblin, (Doob [2]), must hold:

There exists a (finite valued) measure ϕ of sets $A \in \mathcal{F}_X$ with $\phi(X) > 0$, an integer $\nu \geq 1$ and an $\varepsilon > 0$ such that $\pi^{(\nu)}(u,A) \leq 1 - \varepsilon$ if $\phi(A) \leq \varepsilon$.

A set E is a consequent of u_0 if $\pi^{(n)}(u_0,E) = 1$ for all n. A set which is a consequent of every one of its points is called invariant. The set is said to be minimal invariant if it contains no invariant set of smaller ϕ measure.

The theorem we are seeking, as stated by Doob, is:

Theorem 5. Suppose Doeblin's hypothesis holds. Let $D_1, D_2, \ldots$ be the distinct non-null minimal invariant sets (with respect to ϕ). Then for every $u \in X$

$$\lim_{n \to \infty} \pi^{(n)}(u, \bigcup_j D_j) = 1 .$$

We now exhibit a Φ for our process which satisfies Doeblin's hypothesis. Let Φ_1 be a measure on $[0,B]$ defined by

$$\Phi_1([0,0]) = \Phi_1([B,B]) = 1 ,$$

and if $0 \notin A$, $B \notin A$, $\Phi_1(A) = 0$. Let Φ_2 be defined as Lebesgue measure. We claim with the choice of measure $\Phi = \Phi_1 + \Phi_2$, Doeblin's hypothesis will be satisfied for the following cases.

Case a) $F(B) < 1$.

Since $\Phi([0,0]) = 1$, consider a set A, $0 \notin A$, $A \subseteq (0,B]$. Then

$$\pi(u,A) \leq \pi(u,(0,B]) = F(u-) \leq F(B-) = 1 - \varepsilon .$$

Hence, whenever $\Phi(A) \leq \varepsilon$, $\pi(u,A) \leq 1 - \varepsilon$, and Doeblin's hypothesis is satisfied with $\nu = 1$.

Theorem 3'. If $F(B) < 1$, $p^{(1)}(b,B) \equiv 0$.

Proof: The proof rests on the fact that 0 is the only minimal invariant set, since for $A \subseteq (0,B]$, $\pi(u,A) \leq 1 - \varepsilon$, for all u; and for A such that $[0,0] \subset A$, A cannot be minimal invariant since $[0,0]$ is invariant,

($\pi^{(n)}(0,[0,0]) = 1$, for all n), and of smaller Φ measure. Hence by Theorem 5, $\lim_{n\to\infty} \pi^{(n)}(b,[0,0]) = 1$ which implies $p^{(1)}(b,B) \equiv 0$.

Case b). $F(B) = 1$, G absolutely continuous: $dG = g\,dx$ and $\max_x g = k < \infty$.

Let

$$\pi_0(u,x) = \int_{[u-x,u)} dF(t)\; g(t+x-u)\;, \qquad 0 < x \leq u < B$$

$$(12) \qquad = \int_{[0,u)} dF(t)\; g(t+x-u)\;, \qquad u \leq x < B$$

and

$$\pi(u,A) = \int_A \pi_0(u,x)\,dx\;, \qquad 0 \notin A,\; B \notin A\;.$$

Let

$$\pi(u,[0,0]) = 1 - F(u-)$$

$$\pi(u,[B,B]) = \int_{[0,u)} dF(t)\;(1 - G(B - (u-t)))\;.$$

$\pi(u,A)$ when extended to all sets of $\mathcal{F}_X$ is a stochastic transition function agreeing with the function defined in (11). Now

$$\pi_0(\cdot,\cdot) \leq \max_x g(x) = k .$$

Thus for all sets A such that $0 \notin A$, $B \notin A$,

$$\pi(u,A) \leq k\phi(A) .$$

Let the ε of Doeblin's hypothesis be chosen as $\frac{1}{k+1}$. Hence if $\phi(A) \leq \frac{1}{k+1}$,

$$\pi(u,A) \leq \frac{k}{k+1} = 1 - \frac{1}{k+1} .$$

We need not be concerned with sets containing 0 or B since in that case the ϕ measure of the sets would be $\geq 1 > \frac{1}{k+1}$ for $k > 0$.

The following lemma is used in the main theorem of this section.

Lemma 2. If $P(\xi(\omega) - \eta(\omega) > 0) > 0$, then there exists an ε such that $P(\xi(\omega) - \eta(\omega) > \varepsilon) > 0$.

Proof: Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be such that $\varepsilon_n > 0$ and $\varepsilon_n \downarrow 0$. Then the event

$$\{\xi - \eta > 0\} = \bigcup_n \{\xi - \eta > \varepsilon_n\} .$$

Now

$$0 < P(\xi - \eta > 0) \le \sum_n P(\xi - \eta > \varepsilon_n)$$

by the subadditivity of P. Hence there exists an ε such that $P(\xi - \eta > \varepsilon) > 0$.

Theorem 4'. If $F(B) = 1$, $dG = g\,dx$ where $\max\limits_x g = k < \infty$, and if $P(\xi - \eta > 0) > 0$, then $p^{(1)}(b,B) \equiv 0$.

Proof: Let $A \subseteq (0,B]$. Let $\underline{u} \equiv \operatorname{glb}\limits_{u \in A} u$. If $\underline{u} \in A$ choose $u_0 = \underline{u}$, and if $\underline{u} \notin A$, choose $u_0 \in A$ such that $u_0 - u < \varepsilon$. We can always do this since $\underline{u}$ is the glb. The event

$$\{x_1(\omega) = u_0,\ \xi_1 - \eta_1 > \varepsilon\} \subseteq \{x_1(\omega) = u_0,\ x_2(\omega) \subseteq [0,\underline{u})\}$$

Hence assuming the process starts at u_0

$$0 < P(x_1 = u_0)\ P(\xi_1 - \eta_1 > \varepsilon) \le P(x_1 = u_0)\ \pi(u_0[0,\underline{u}))$$

which implies $\pi(u_0,[0,\underline{u})) > 0$ and hence $\pi(u_0,A) < 1$. Thus A cannot be invariant.

If $[0,0] \subset A$, A cannot be minimal invariant since $[0,0]$ is invariant and certainly of smaller Φ measure. Hence $[0,0]$ is the only minimal invariant set, and thus by Theorem 5, $p^{(1)}(b,B) \equiv 0$.

4. THE (α,b,B) PROCESS

The (α,b,B) process is one in which we consider service to terminate with probability $0 \leq \alpha \leq 1$ if an item reaches the barrier and to continue with probability $1-\alpha$, the server retracting to the next element in queue. Let $p_n^{(\alpha)}(b,B)$, $\pi_n^{(\alpha)}(b,B)$, $p^{(\alpha)}(b,B)$ be the analogs of the probabilities of the (1,b,B) process. Let

$$p^{(\alpha)}(x,b,B) = \sum p_n^{(\alpha)}(b,B)x^n .$$

We have the following relationship between $p^{(\alpha)}$ and $p^{(1)}$.

Theorem 6. If $\alpha > 0$,

$$p^{(\alpha)}(x,b,B) = \frac{\alpha p^{(1)}(x,b,B)}{1-(1-\alpha)\int_{[0,\infty)} p^{(1)}(x,t,B)\, dG_0(t)}$$

where

$$\begin{aligned} G_0 &= G(x) , & x \leq B \\ &= 1 - G(B) , & x > B. \end{aligned}$$

Proof: In the (α,b,B) process, for $k \geq 0$ let

$A_0(b) = \{0 \text{ elements are serviced}\}$

$A_k(b) = \{\text{elements } E_i,\ i = 1,\dots k \text{ are serviced, } E_{k+1} \text{ is not}\}$,

B_{k+1} = {server is disabled at arrival of E_{k+1} at barrier},

D_k = {exactly k elements are serviced},

and

$C_{k,n}$ = {exactly n-k of the elements in the set $\{E_i\}_{i=k+2}^{\infty}$ are serviced}.

D_n can be written as the disjoint union

$$D_n = A_n\, B_{n+1} \cup \bigcup_{k=0}^{n} A_k\, \overline{B}_{k+1}\, C_{k,n}$$

where $\overline{B}_{k+1}$ is the complement of B_{k+1}. Thus

$$(13)\quad P(D_n) = p_n^{(\alpha)}(b) = P(A_n)\; P(B_{n+1} \mid A_n)$$

$$+ \sum P(A_k)\; P(\overline{B}_{k+1} \mid A_k)\; P(C_{k,n} \mid A_k \overline{B}_{k+1}) \,.$$

The quantities $\{P(A_k(b))\}$ satisfy the recursion relation (1) with the same initial condition. Hence

$$P(A_k(b)) \;=\; p_k^{(1)}(b,B) \,.$$

We have using a conditional argument

$$P(C_{k,n}\; A_k\; \overline{B}_{k+1}) = \int_{[0,\infty)} p_{n-k}^{(\alpha)}(x,B)\; dG_0(x)$$

and clearly

$$P(B_{n+1} | A_n) = \alpha \qquad \text{for all } n$$

(13) becomes

$$(14) \quad p_n^{(\alpha)}(b,B) = \alpha p_n^{(1)}(b,B) + (1-\alpha) \sum_{k=0}^{n} p_k^{(1)}(b,B) \cdot \int_{[0,\infty)} p_{n-k}^{(\alpha)}(x,B)\, dG_0(x) .$$

Introducing generating functions in (14), we get

$$(15) \quad p^{(\alpha)}(x,b,B) = \alpha p^{(1)}(x,b,B) + (1-\alpha) p^{(1)}(x,b,B) U^{(\alpha)}(x)$$

where

$$U^{(\alpha)}(x) = \sum_{j=0}^{\infty} \int_{[0,\infty)} p_j^{(\alpha)}(s,B)\, dG_0(s)\, x^j .$$

By the monotone convergence theorem we have

$$\int_{[0,\infty)} p^{(\alpha)}(x,b)\, dG(b) = \int_{[0,\infty)} \sum_{n=0}^{\infty} p_n^{(\alpha)}(b,B) x^n\, dG_0(b)$$

$$= \sum_{n=0}^{\infty} \int_{[0,\infty)} p_n^{(\alpha)}(b,B)\, dG_0(b) x^n = U^{(\alpha)}(x).$$

Thus (15) leads to

$$U^{(\alpha)}(x) = \alpha\, U^{(1)}(x) + (1-\alpha)U^{(1)}(x)\, U^{(\alpha)}(x) .$$

Since $\alpha > 0$

$$(16)\quad U^{(\alpha)}(x) = \frac{\alpha\, U^{(1)}(x)}{1-(1-\alpha)U^{(1)}(x)} = \frac{\alpha \int_{[0,\infty)} p^{(1)}(x,b,B)\, dG_0(b)}{1 - (1-\alpha)\int_{[0,\infty)} p^{(1)}(x,b)\, dG_0(b)}$$

Substituting (16) into (15), we get the desired result.

Corollary 6.1. If $p^{(1)}(1,b,B) \equiv 1$, $p^{(\alpha)}(1,b,B) \equiv 1$, $\alpha > 0$.

Proof: The proof is immediate by substitution.

Corollary 6.1 enables us to state that since

$$p^{(\alpha)}(1,b,B) = 1 - p^{(\alpha)}(b,B) , \qquad \alpha > 0$$

whenever $p^{(1)}(b,B) \equiv 0$, $p^{(\alpha)}(b,B) = 0$, $\alpha > 0$. Hence Theorems 3, 3' and 4, 4' for $\alpha > 0$ can be extended to the (α,b,B) process.

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13. ABSTRACT

An extension of a model considered by Hanisch and Hirsch [Comm. Pure Appl. Math., 11/63] is investigated in which the server is of "bounded range." The properties of $p^{(1)}(b,B)$, a measure of effectiveness of the server relative to the system, are explored. In particular conditions are given for the case of an ineffective server, $(p^{(1)}(b,B) = 0)$. The model and its properties are further extended to a "partial absorbing barrier."

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